ELSEVIER

Journal of Computational and Applied Mathematics 91 (1998) 97–105

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

On the limit relations between classical continuous and discrete orthogonal polynomials

E. Godoy[a,*], A. Ronveaux[b], A. Zarzo[c], I. Area[a]

[a] *Departamento de Matemática Aplicada, Escuela Técnica Superior de Ingenieros Industriales y Minas, Universidad de Vigo, 36200-Vigo, Spain*

[b] *Mathematical Physics, Facultés Universitaires Notre-Dame de la Paix, B-5000 Namur, Belgium*

[c] *Departamento de Matemática Aplicada, Escuela Técnica Superior de Ingenieros Industriales, Universidad Politécnica de Madrid, c/ José Gutiérrez Abascal 2, 28006 Madrid, Spain*

Received 1 May 1997

Abstract

Limit relations between classical continuous (Jacobi, Laguerre, Hermite) and discrete (Charlier, Meixner, Kravchuk, Hahn) orthogonal polynomials are well known and can be described by relations of type $\lim_{\lambda\to\infty} P_n(x;\lambda) = Q_n(x)$. Deeper information on these limiting processes can be obtained from the expansion $P_n(x;\lambda) = \sum_{k=0}^{\infty} R_k(x;n)/\lambda^k$. In this paper a method for the recursive computation of coefficients $R_k(x;n)$ is designed being the main tool the consideration of a closely related connection problem which can be solved, also recurrently, by using an algorithm recently developed by the authors. © 1998 Elsevier Science B.V. All rights reserved.

AMS classification: 33C25

Keywords: Orthogonal polynomials; Connection problem; Limit relation

At the lowest level of the Askey tableau [1, 6] of hypergeometric polynomials, one can find the so-called classical continuous (Jacobi, Laguerre, Hermite) and discrete (Charlier, Meixner, Kravchuk, Hahn) orthogonal polynomials which can be written in terms of certain ${}_pF_q$ hypergeometric functions. An interesting aspect of this tableau are the limit transitions between some of these polynomial families [6, 12], which are relevant in several problems appearing in Mathematical Physics and Quantum Mechanics such as contractions of linear groups or asymptotics for Clebsch–Gordan coefficients (see e.g. [16, 17]).

These limit transitions can be described by relations of type

$$\lim_{\lambda\to\infty} P_n(x;\lambda) = Q_n(x) \tag{1}$$

* Corresponding author. E-mail: egodoy@dma.uvigo.es.

PII S0377-0427(98)00026-0

being λ the limiting parameter and $\{P_n(x;\lambda)\}$ and $\{Q_n(x)\}$ the classical families connected by the limit property. Of course, both polynomial families could depend on some extra parameters not affected by the limiting process.

On the other hand, it turns out that in the classical situation we shall deal with here, the polynomial $P_n(x;\lambda)$ in (1) is always a rational function of λ such that there exists an expansion of the form

$$P_n(x;\lambda)=\sum_{k=0}^{\infty}\frac{R_k(x;n)}{\lambda^k}. \tag{2}$$

Clearly, the only information provided by the limit relation (1) on the latter expansion is $R_0(x;n)=Q_n(x)$.

It is our main aim here to go further on computing coefficients $R_k(x;n)$ in expansion (2). This can be done by considering the so-called connection problem, i.e.

$$P_n(x;\lambda)=\sum_{m=0}^{n}C_m(n;\lambda)Q_m(x) \tag{3}$$

between the two classical families involved in the limit relation (1). Notice that, if the coefficients $S_k(n;m)$ in the expansion

$$C_m(n;\lambda)=\sum_{k=0}^{\infty}\frac{S_k(n;m)}{\lambda^k} \tag{4}$$

are known, then one obviously has

$$R_k(x;n)=\sum_{m=0}^{n}S_k(n;m)Q_m(x) \tag{5}$$

showing that coefficients $R_k(x;n)$ can be expressed as a linear combination of polynomials belonging to the limit family $\{Q_n(x)\}$.

The specific set of limit relations (1) to be treated together with the corresponding connection problems (3) are listed in Tables 1 (continuous case) and 2 (discrete case). Notations for the classical continuous and discrete families are summarized in these two tables and coincide with the ones commonly used in the literature (see e.g. [4, 8, 9, 15]). Concerning normalization, from now on monic polynomials will be considered.

Having in mind expressions (4) and (5), for computing coefficients $R_k(x;n)$ in (2) we first use an algorithm, recently developed by the authors [2, 5, 13–15, 19], which provides a recurrence relation for the connection coefficients $C_m(n;\lambda)$ in (3) allowing us to compute (also recursively) the coefficients $S_k(n;m)$ in expansion (4). This algorithm can be summarized as follows. Assume that:

(a) The polynomial $P_n(x;\lambda)$ in (3) satisfies a difference or differential equation

$$\mathscr{L}_r[P_n(x;\lambda)]:=\sum_{i=0}^{r}A_i(x;n)\mathscr{D}^iP_n(x;\lambda)=0,$$

where $\mathscr{D}$ stands for the difference ($\mathscr{D}f(x)=f(x+1)-f(x)$) or derivative ($\mathscr{D}f(x)=f'(x)$) operator and the coefficients $A_i(x;n)$ are polynomials in x of fixed degree (i.e. the degree does not depend on n).

Table 1
Monic classical continuous orthogonal polynomials: limit transitions and related connection problems

Limiting relation	Related connection problem
Jacobi: $P_n^{(a,b)}(x)$ $(a,b>-1)\to$ Laguerre: $L_n^{(a)}(x)$ $(a>-1)$	
$\lim_{b\to\infty}\left(-\frac{b}{2}\right)^n P_n^{(a,b)}\left(1-\frac{2x}{b}\right)=L_n^{(a)}(x)$	$\left(-\frac{b}{2}\right)^n P_n^{(a,b)}\left(1-\frac{2x}{b}\right)=\sum_{m=0}^{n} C_m(n,a,b)L_m^{(a)}(x)$
Gegenbauer: $P_n^{(c,c)}(x)$ $(c>-1)\to$ Hermite: $H_n(x)$	
$\lim_{c\to\infty} c^n P_n^{(c^2,c^2)}\left(\frac{x}{c}\right)=H_n(x)$	$c^n P_n^{(c^2,c^2)}\left(\frac{x}{c}\right)=\sum_{m=0}^{n} C_m(n,c)H_m(x)$
Laguerre: $L_n^{(d)}(x)$ $(d>-1)\to$ Hermite: $H_n(x)$	
$\lim_{d\to\infty}\frac{L_n^{(d^2)}(d^2+d\sqrt{2}x)}{(2d^2)^{n/2}}=H_n(x)$	$\frac{L_n^{(d^2)}(d^2+d\sqrt{2}x)}{(2d^2)^{n/2}}=\sum_{m=0}^{n} C_m(n,d)H_m(x)$

(b) The family $\{Q_m(x)\}$ in (3) verifies a finite $(h+2)$-term recurrence relation

$$xQ_m(x)=\sum_{k=m-h}^{m+1} B_{m,k}Q_k(x), \tag{6}$$

where $B_{m,k}$ are x-independent coefficients. Moreover, this Q_m-family also verifies a finite structure relation with respect to the operator $\mathscr{D}$,

$$p(x)\mathscr{D}Q_m(x)=\sum_{k=m-s-1}^{m+t-1} F_{m,k}Q_k(x), \tag{7}$$

where $F_{m,k}$ are constants, s is a fixed integer and $p(x)$ is a t-degree polynomial.

Then, the action of the rth-order difference operator $\mathscr{L}_r$ on both sides of the connection problem (3), gives rise to the relation:

$$\sum_{m=0}^{n} C_m(n;\lambda)\mathscr{L}_r[Q_m(x)]=0 \tag{8}$$

which contains terms of the form $x^j\mathscr{D}^iQ_m(x)$, where i runs from 0 to r and j depends upon the degree of the polynomial coefficients characterizing the operator $\mathscr{L}_r$. The appropriate (and possibly repeated) use of properties (6) and (7) allows us to express all terms appearing in the latter sum as a linear combination (with constant coefficients) of the polynomials $Q_k(x)$ themselves. Thus, if the family $\{Q_m(x)\}$ satisfies (6) and (7), it is always possible to transform (8) into a relation of the

Table 2
Monic classical discrete orthogonal polynomials: limit transitions and related connection problems

Limiting relation	Related connection problem
Hahn: $\left\{\begin{matrix} H_n^{(a,b)}(x;N),\ n\leqslant N-1, \\ a,b>-1,\ N\in\mathbb{Z}^+. \end{matrix}\right\}$	$\to$ Kravchuk: $\left\{\begin{matrix} K_n^{(a)}(x;N),\ n\leqslant N, \\ a>-1,\ N\in\mathbb{Z}^+. \end{matrix}\right\}$
$\lim_{\beta\to\infty} H_n^{((1-\alpha)\beta,\alpha\beta)}(x;N)$ $=K_n^{(\alpha)}(x;N-1)$	$H_n^{((1-\alpha)\beta,\alpha\beta)}(x;N)$ $=\sum_{m=0}^{n} C_m(n,N,\alpha,\beta)K_n^{(\alpha)}(x;N-1)$
Hahn: $\left\{\begin{matrix} H_n^{(a,b)}(x;N),\ n\leqslant N-1, \\ a,b>-1,\ N\in\mathbb{Z}^+. \end{matrix}\right\}$	$\to$ Meixner: $M_n^{(a,b)}(x)$ $(a,b>-1)$
$\lim_{N\to\infty} H_n^{(N/(\delta-1),\gamma-1)}(x;N)$ $=M_n^{(\gamma,\delta)}(x)$	$H_n^{(N/(\delta-1),\gamma-1)}(x;N)$ $=\sum_{m=0}^{n} C_m(n,N,\gamma,\delta)M_n^{(\gamma,\delta)}(x)$
Kravchuk: $\left\{\begin{matrix} K_n^{(a)}(x;N),\ n\leqslant N, \\ a>-1,\ N\in\mathbb{Z}^+. \end{matrix}\right\}$	$\to$ Charlier: $C_n^{(a)}(x)$ $(a>-1)$
$\lim_{N\to\infty} K_n^{(\eta/N)}(x;N)$ $=C_n^{(\eta)}(x)$	$K_n^{(\eta/N)}(x;N)$ $=\sum_{m=0}^{n} C_m(n,N,\eta)C_n^{(\eta)}(x)$
Meixner: $M_n^{(a,b)}(x)$ $(a,b>-1)$	$\to$ Charlier: $C_n^{(a)}(x)$ $(a>-1)$
$\lim_{\mu\to\infty} M_n^{(\mu,\nu/(\mu+\nu))}(x)$ $=C_n^{(\nu)}(x)$	$M_n^{(\mu,\nu/(\mu+\nu))}(x)$ $=\sum_{m=0}^{n} C_m(n,\mu,\nu)C_n^{(\nu)}(x)$

form: $\sum_{m=0}^{K} C_m(n;\lambda)R_m[Q_m(x)]=0$. Here, K is a positive integer whose specific value depends on the operator $\mathscr{L}_r$ and also on the relations (6) and (7) (see [13]), and R_m denotes a linear operator with constant (x-independent) coefficients acting on the index m. Then, a simple shift of indexes leads to the expression

$$\sum_{m=0}^{K} M_m[C_m(n;\lambda)]Q_m(x)=0 \quad \text{then } M_m[C_m(n;\lambda)]=0 \quad (m=0,\ldots,K).$$

Here M_m denotes a linear operator with constant coefficients acting on m. So, in this way, a linear system of equations satisfied by the connection coefficients is obtained. Finally, due to its particular structure (see [13]), from this linear system a recurrence relation (in the index m only) for the $C_m(n;\lambda)$-coefficients can be easily devised.

Clearly, polynomials involved in the seven connection problems listed in Tables 1 and 2 verify the above requirements (a) and (b). Moreover, for all of them, the family $\{Q_m(x)\}$ is classical (continuous or discrete) and so it also satisfies a relation of type

$$Q_m(x) = E_m \mathscr{D} Q_{m+1}(x) + F_m \mathscr{D} Q_m(x) + G_m \mathscr{D} Q_{m-1}(x),$$

called difference or derivative representation, which is a consequence of the orthogonality of the $\{\mathscr{D}Q_n(x)\}$ polynomial family [7, 9]. As shown in [2, 5], the existence of this representation allows to develop a modification of the above-described algorithm which sometimes provides shorter recurrences. Explicit expressions of these recurrences for the seven connection problems here considered has been included in an appendix at the end of the manuscript.

First consequence of those recurrences is the possibility of computing asymptotic behaviour of the connection coefficients with respect to the limiting parameter (λ in Eq. (1)), i.e. to give explicitly the number $h(n,m)$ in the expression:

$$C_m(n;\lambda) \sim \left(\frac{1}{\lambda}\right)^{h(n,m)} S_{h(n,m)}(n;m) \quad (\lambda \to \infty), \tag{9}$$

where the symbol "$\sim$" is used in the sense of Olver [10] (i.e. $\phi(x) \sim \psi(x)$ as $x \to \infty$ means $\lim_{x\to\infty}(\phi(x)/\psi(x)) = 1$) and (as already mentioned) coefficients $S_{h(n,m)}(n;m)$ defined by Eq. (4) can be obtained recursively. The value of numbers $h(n,m)$ for the connection problems here considered are listed in Table 3. Computations involved in their obtention are rather cumbersome. They have been performed with the help of *Mathematica* symbolic language [18].

It is remarkable that, as Table 3 shows, the asymptotic behaviour is the same in all cases, except when the limit polynomial belongs to the Hermite family where a slight difference appears coming

Table 3
Asymptotic behaviour of connection coefficients (appearing in Tables 1 and 2) with respect to the corresponding limiting parameter. First column identifies the concrete problem and second gives the value of the number $h(n,m)$ in Eq. (9). Notations are those of Tables 1 and 2 and $E[x]$ stands for the largest integer not exceeding x

Connection Problem	Number $h(n,m)$ in Eq. (9)
Jacobi $\to$ Laguerre	$h(n,m) = E[\frac{n-m+1}{2}]$
Gegenbauer $\to$ Hermite	$h(n,m) = \begin{cases} 0 & \text{for } n-m=2p+1 \\ E[\frac{n-m+2}{4}] & \text{for } n-m=2p \end{cases}$
Laguerre $\to$ Hermite	$h(n,m) = \begin{cases} 2E[\frac{p+1}{3}]+1 & \text{for } n-m=2p+1 \\ 2E[\frac{p+2}{3}] & \text{for } n-m=2p \end{cases}$
Hahn $\to$ Kravchuk	$h(n,m) = E[\frac{n-m+1}{2}]$
Hahn $\to$ Meixner	$h(n,m) = E[\frac{n-m+1}{2}]$
Kravchuk $\to$ Charlier	$h(n,m) = E[\frac{n-m+1}{2}]$
Meixner $\to$ Charlier	$h(n,m) = E[\frac{n-m+1}{2}]$

Table 4
Coefficients $R_1(x;n)$ in expansion (2) for all classical continuous and discrete orthogonal polynomials on the left-hand side of limiting relations listed in Tables 1 and 2. In each case the expanding parameter (λ in Eq. (2)) corresponds to the limiting one in those tables, whose notations are also used

Expansion (2)	$R_1(x;n)$
Jacobi $\left(-\frac{b}{2}\right)^n P_n^{(a,b)}\left(1-\frac{2x}{b}\right)=\sum_{k=0}^{\infty}\frac{R_k(x;n)}{b^k}$	$\frac{n(n+a)}{2}[2(2n+a)L_{n-1}^{(a)}(x)$ $+(n-1)(n+a-1)L_{n-2}^{(a)}(x)]$
Gegenbauer $c^n P_n^{(c^2,c^2)}\left(\frac{x}{c}\right)=\sum_{k=0}^{\infty}\frac{R_k(x;n)}{c^k}$	$\frac{n(n-1)}{24}[3(2n-1)H_{n-2}(x)$ $+(n-2)(n-3)H_{n-4}(x)]$
Laguerre $\frac{L_n^{(d^2)}(d^2+d\sqrt{2}x)}{(2d^2)^{n/2}}=\sum_{k=0}^{\infty}\frac{R_k(x;n)}{d^k}$	$-\frac{n}{6\sqrt{2}}[6nH_{n-1}(x)$ $+(n-1)(n-2)H_{n-3}(x)]$
Hahn $H_n^{((1-\alpha)\beta,\alpha\beta)}(x;N)=\sum_{k=0}^{\infty}\frac{R_k(x;n)}{\beta^k}$	$\frac{n(n-N)}{2}[2n(1-2\alpha)K_{n-1}^{(\alpha)}(x;N-1)$ $+(n-1)(n-N+1)\alpha(\alpha-1)$ $K_{n-2}^{(\alpha)}(x;N-1)]$
Hahn $H_n^{(N/(\delta-1),\gamma-1)}(x;N)=\sum_{k=0}^{\infty}\frac{R_k(x;n)}{N^k}$	$\frac{n\delta(n+\gamma-1)}{2(\delta-1)^2}$ $\times[2(n+\delta(n+\gamma-1))M_{n-1}^{(\gamma,\delta)}(x)$ $+\delta^2(n+\gamma-1)(n-1)(n+\gamma-2)$ $\times(3n-2+\delta(3n+3\gamma-5))M_{n-2}^{(\gamma,\delta)}(x)]$
Kravchuk $K_n^{(\eta/N)}(x;N)=\sum_{k=0}^{\infty}\frac{R_k(x;n)}{N^k}$	$\frac{\eta n(n-1)}{2}[2C_{n-1}^{(\eta)}(x)+\eta C_{n-2}^{(\eta)}(x)]$
Meixner $M_n^{(\mu,\nu/(\mu+\nu))}(x)=\sum_{k=0}^{\infty}\frac{R_k(x;n)}{\mu^k}$	$\frac{\nu n(1-n)}{2}[2C_{n-1}^{(\nu)}(x)+\nu C_{n-2}^{(\nu)}(x)]$

from the symmetry of this classical polynomial sequence. Moreover, it is clear that limit

$$\lim_{\lambda\to\infty} C_m(n;\lambda)=\begin{cases}0 & \text{if } m<n,\\ 1 & \text{if } m=n\end{cases}$$

always holds and so one has the identity $R_0(x;k)=Q_n(x)$ (see Eq. (2)), as it should be expected.

Now, we are able to go further on computing coefficients $R_k(x;n)$ in expansion (2) by summing up contributions of each $C_m(n;\lambda)$-coefficient to $(1/\lambda)^k$. Clearly, these contributions can be obtained recursively from the recurrence relations listed in the appendix. As illustration we consider the simplest case, i.e. $k=1$. The corresponding coefficients $R_1(x;n)$ are given in Table 4 for all limiting relations here studied.

Finally, let us remark an interesting property which Table 4 shows: Coefficients $R_1(x;n)$ are always linear combinations of two polynomials belonging to the corresponding limit classical family. As it is well known [3, 4, 11], this means that $R_1(x;n)$ is always a quasi-orthogonal polynomial of order one, except in the two Hermite cases where the quasi-orthogonality is of order two.

Acknowledgements

This work has been partially supported by NATO grant CRG-960213. The work of E.G. and I.A. has been partially finantial supported by Xunta de Galicia-Universidade de Vigo under grant 645021703. A.Z. also wishes to acknowledge partial finantial support from DGES (Ministerio de Educación y Cultura of Spain) under contract PG95-1205.

Appendix. Recurrence relations for connection coefficients appearing in Tables 1 and 2

Application of the algorithm developed in [2, 5] to the seven connection problems listed in Tables 1 and 2 gives rise to the following recurrence relations for the corresponding connection coefficients. Notations are the same as in those tables and $(A)_k = A(A+1)\cdots(A+k-1)$, $(A)_0 = 1$, will stand for the well-known Pochhammer symbol.

Jacobi → Laguerre

- Limiting parameter for relation (1): $\lambda = b$.
- Third-order recurrence relation for the connection coefficients:

$$\begin{aligned}&-(m)_3(m+a+1)_2 C_{m+2}(n,a,b)\\&-(m)_2(m+a+1)(2(1+a)+3m)C_{m+1}(n,a,b)\\&+m(-a^2-m(1+4a+b+3m)+n(1+a+b+n))C_m(n,a,b)\\&+(1-m+n)(a+b+m+n)C_{m-1}(n,a,b)=0 \quad (1\leqslant m\leqslant n).\end{aligned}$$

- Initial conditions: $C_{n+2}(n,a,b)=C_{n+1}(n,a,b)=0$, $C_n(n,a,b)=1$.

Gegenbauer → Hermite

- Limiting parameter for relation (1): $\lambda = c$.
- Fourth-order recurrence relation for the connection coefficients:

$$\begin{aligned}&4(n-m+2)(m+n+2c-1)C_{m-2}(n,c)\\&\quad +2(m-1)_2(2m-1)C_m(n,c)-(m-1)_4 C_{m+2}(n,c)=0 \quad (2\leqslant m\leqslant n+1).\end{aligned}$$

- Initial conditions: $C_{n+3}(n,c)=C_{n+2}(n,c)=C_{n+1}(n,c)=0$, $C_n(n,c)=1$.

Laguerre → Hermite

– Limiting parameter for relation (1): $\lambda = d$.
– Third-order recurrence relation for the connection coefficients:

$$4d^2(2-m+n)C_{m-2}(n,d) + 2\sqrt{2}|d|(m-1)^2 C_{m-1}(n,d) + \sqrt{2}|d|(m-1)_3 C_{m+1}(n,d) = 0 \quad (2 \leqslant m \leqslant n+1).$$

– Initial conditions: $C_{n+2}(n,d) = C_{n+1}(n,d) = 0$, $C_n(n,d) = 1$.

Hahn → Kravchuk

– Limiting parameter for relation (1): $\lambda = \beta$.
– Third-order recurrence relation for the connection coefficients:

$$\begin{aligned}&\alpha(\alpha-1)_2 m_3 (N-2-m)_2 C_{m+2}(n,N,\alpha,\beta) + \alpha m(1+m)(N-1-m)\\ &\quad\times(-1+2\alpha-2m+3\alpha m+N-\alpha N)C_{m+1}(n,N,\alpha,\beta)\\ &\quad+m\{\alpha[m(3m-2N+1)-n(n+1)+\beta(m-n)]+m(N-m)\}\\ &\quad\times C_m(n,N,\alpha,\beta)+(1-m+n)(m+n+\beta)C_{m-1}(n,N,\alpha,\beta)=0 \quad (1 \leqslant m \leqslant n).\end{aligned}$$

– Initial conditions: $C_{n+2}(n,N,\alpha,\beta) = C_{n+1}(n,N,\alpha,\beta) = 0$, $C_n(n,N,\alpha,\beta) = 1$.

Hahn → Meixner

– Limiting parameter for relation (1): $\lambda = N$.
– Third-order recurrence relation for the connection coefficients:

$$\begin{aligned}&\delta^3 m_3(\gamma+m)_2 C_{m+2}(n,N,\gamma,\delta)\\ &\quad+(1-\delta)\delta^2 m_2(\gamma+m)(\gamma+\gamma\delta+2m+\delta m)C_{m+1}(n,N,\gamma,\delta)\\ &\quad+(-1+\delta)^2\delta m(\delta-2\gamma\delta+\gamma^2\delta-m+\gamma m-2\delta m+3\gamma\delta m+m^2\\ &\quad+2\delta m^2-\gamma\delta n-\delta n^2+mN-\delta mN-nN+\delta nN)C_m(n,N,\gamma,\delta)\\ &\quad+(-1+\delta)^3(1-m+n)(-\delta+\gamma\delta+\delta m+\delta n+N-\delta N)\\ &\quad\times C_{m-1}(n,N,\gamma,\delta)=0 \quad (1 \leqslant m \leqslant n).\end{aligned}$$

– Initial conditions: $C_{n+2}(n,N,\gamma,\delta) = C_{n+1}(n,N,\gamma,\delta) = 0$, $C_n(n,N,\gamma,\delta) = 1$.

Kravchuk → Charlier

– Limiting parameter for relation (1): $\lambda = N$.
– Second-order recurrence relation for the connection coefficients (see [2]):

$$\eta^2 m(m+1)C_{m+1}(n,N,\eta) + \eta m(m-1)C_m(n,N,\eta) + N(m-n-1)C_{m-1}(n,N,\eta) = 0 \quad (1 \leqslant m \leqslant n).$$

– Initial conditions: $C_{n+1}(n,N,\eta) = 0$, $C_n(n,N,\eta) = 1$.

Meixner $\to$ *Charlier*

- Limiting parameter for relation (1): $\lambda = \mu$.
- Second-order recurrence relation for the connection coefficients (see [2]):

$$\begin{aligned} &v^2 m(m+1)C_{m+1}(n,\mu,v) + vm(m-1)C_m(n,\mu,v) \\ &\quad +\mu(n-m+1)C_{m-1}(n,\mu,v) = 0 \quad (1 \leqslant m \leqslant n). \end{aligned}$$

- Initial conditions: $C_{n+1}(n,\mu,v)=0$, $C_m(n,\mu,v)=1$.

References

[1] G.E. Andrews, R. Askey, Classical orthogonal polynomials, in: C. Brezinski et al. (eds.), Polynômes Orthogonaux et Applications, Lecture Notes in Math., Vol. 1171, Springer, Berlin, 1985, pp. 36–62.
[2] I. Area, E. Godoy, A. Ronveaux, A. Zarzo, Minimal recurrence relation for connection coefficients between classical orthogonal polynomials: discrete case. J. Comput. Appl. Math. 89 (1998) 309–325.
[3] A. Branquinho, F. Marcellán, Generating new classes of orthogonal polynomials, Internat. J. Math. Math. Sci. 19 (1996) 643–656.
[4] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[5] E. Godoy, I. Area, A. Ronveaux, A. Zarzo, Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: continuous case. J. Comput. Appl. Math. 84 (1997) 257–275.
[6] R. Koekoek, R.F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Fac. Techn. Math. Informatics, Delft University of Technology, Report 94-05, Delft 1994.
[7] F. Marcellán, A. Branquinho, J. Petronilho, Classical orthogonal polynomials: a functional approach, Acta Appl. Math. 34 (1994) 283–303.
[8] A.F. Nikiforov, S.K. Suslov, V.B. Uvarov, Classical Orthogonal Polynomials of a Discrete Variable, Springer, Berlin, 1991.
[9] A.F. Nikiforov, V.B. Uvarov, Special Functions of Mathematical Physics, Birkhäuser, Basel, 1988.
[10] F.J.W. Olver, Asymptotics and Special Functions, Academic Press, New York and London, 1974.
[11] A. Ronveaux, Quasi-orthogonality and differential equations, J. Comput. Appl. Math. 29 (1990) 243–248.
[12] A. Ronveaux, Limit transitions in the Askey tableau, in: W. Van Assche (Ed.), Open Problems Section J. Comput. Appl. Math. 48 (1–2) (1993) 234–236.
[13] A. Ronveaux, A. Zarzo, E. Godoy, Recurrence relation for connection coefficients between two families of orthogonal polynomials, J. Comp. Appl. Math. 62 (1995) 67–73.
[14] A. Ronveaux, S. Belmehdi, E. Godoy, A. Zarzo, Recurrence relation approach for connection coefficients. Application to classical discrete orthogonal polynomials, in: D. Levi, L. Vinet, P. Winternitz (Eds.), Proc. Internat. Workshop on Symmetries and Integrability of Difference Equations, CRM Proc. and Lecture Note Series 9, AMS, Providence, Rhode Island, 1996, pp. 321–337.
[15] G. Szegö, Orthogonal Polynomials A.M.S. Colloquium Publications vol. XXIII, AMS, Providence, Rhode Island, 1978.
[16] J.D. Talman, Special functions. A group theoretic approach. Based on lectures by E.P. Wigner and W.A. Benjamin, INC, New York-Amsterdam, 1968.
[17] N.Ja. Vilenkin, Special Functions and the Theory of Groups Representations, AMS, Providence, Rhode Island, 1968.
[18] Wolfram Research, Inc., Mathematica version 2.2, Wolfram Research Inc., Champaign, Illinois, 1993.
[19] A. Zarzo, I. Area, E. Godoy, A. Ronveaux, Results for some inversion problems for classical continuous and discrete orthogonal polynomials, J. Phys. A 30 (1997) L35–L40.